Maths for Computer Science
Sum of cubes

Denis TRYSTRAM

Motivation of this session

- Develop intuition
- Survey of some proof technics (training)
- Gain experience

The target problem

Sum of n first cubes

$C_n = \sum_{k=1}^{n} k^3$

How to start?

- Take a look at the asymptotic behaviour[1]
- Investigate the first ranks to get an idea

$$C_1 = 1$$
$$C_2 = 1 + 8 = 9$$
$$C_3 = 1 + 8 + 27 = 36$$
$$C_4 = 1 + 8 + 27 + 64 = 100$$
$$C_5 = 1 + 8 + 27 + 64 + 125 = 225$$

- All these values are perfect squares: $1, 3^2, 6^2, 10^2 and 15^2$.

[1] this is easy to show it is in $\Theta(n^4)$

- A more attentive observation evidences a link with the triangular numbers[2] Δ_n
  $1, 3, 6, 10$ and 15.

Proposition

$C_n = \Delta_n^2$

This is a just guess, **not a proof!**

[2] defined as the sum of the first integers

The classical way to solve

Prove by recurrence on n

- **Base case.**
  $C_1 = 1 = 1^2$ is true since $\Delta_1 = 1$
- **Induction step.**
  Assume the proposition is true for n.
  $C_{n+1} = C_n + (n+1)^3$
  $C_{n+1} = \Delta_n^2 + (n+1)^3$ by applying the recurrence hypothesis
  $= \frac{1}{4}(n+1)^2 n^2 + (n+1)^3$
  $= \frac{1}{4}(n+1)^2(n^2 + 4 \cdot (n+1))$
  $= \frac{1}{4}(n+1)^2(n+2)^2 = \Delta_{n+1}^2$

□

Other ways to solve the problem

Let us investigate other directions that will strengthen our understanding of the the mathematical object *Sum of cubes*.

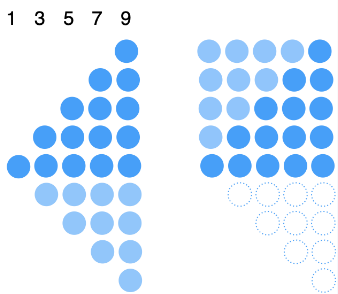
I propose to concentrate on the simplified problem of computing the sum of the first n odd integers.

Sum of odds

The problem. Determine the sum of the first odd integers. denoted by $S_n = \sum_{k=1}^{n}(2k-1)$.

- This result may be established by using Fubini's principle.
- Each integer k is represented by k tokens.
  The arrangement of the tokens gives two ways for counting.

- The second arrangement is clearly a perfect square.



- We deduce: $S_n = n^2$
- The figure gives only the *principle* (for $n = 5$) and should be proved for any n, this is easy by any method and is let to the reader.

Coming back to the original problem on cubes

Proposition:

For all n,

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{\Delta_n} (2k-1) = \Delta_n^2 \qquad (1)$$

- We proved previously
  $\sum_{k=1}^{\Delta_n} (2k-1) = \Delta_n^2$ and
  $\sum_{k=1}^{n} k^3 = \Delta_n^2$
- Thus, by transitivity, for all n, $\sum_{k=1}^{n} k^3 = \sum_{k=1}^{\Delta_n} (2k-1)$

Direct proof of the last equality

We take the odd integers in order and arrange them into groups whose successive sizes increase by 1 at each step, as follows:

$$\begin{array}{lllll} \text{group 1 (size 1):} & 1, & & & \\ \text{group 2 (size 2):} & 3, & 5, & & \\ \text{group 3 (size 3):} & 7, & 9, & 11, & \\ \text{group 4 (size 4):} & 13, & 15, & 17, & 19 \\ \vdots & \vdots & & & \end{array} \tag{2}$$

Features of Table (2)

We observe first that[3] the i elements of the ith group add up to i^3:

$$
\begin{array}{lrrrrl}
\text{group 1 (size 1):} & 1, & & & & : \text{sum} = 1^3 \\
\text{group 2 (size 2):} & 3, & 5, & & & : \text{sum} = 2^3 \\
\text{group 3 (size 3):} & 7, & 9, & 11, & & : \text{sum} = 3^3 \\
\text{group 4 (size 4):} & 13, & 15, & 17, & 19 & : \text{sum} = 4^3
\end{array}
$$

[3] at least within the illustrated portion of the table

We observe next that, by construction, the ith group/row of odd integers in the table consists of the i consecutive odd numbers beginning with the $(\Delta_{i-1} + 1)$th odd number, namely, $2\Delta_{i-1} + 1$.

Since consecutive odd numbers differ by 2, this means that the ith group (for $i > 1$) comprises the following i odd integers:

$$2\Delta_{i-1} + 1,\ 2\Delta_{i-1} + 3,\ 2\Delta_{i-1} + 5,\ \ldots,\ 2\Delta_{i-1} + (2i - 1)$$

Therefore, the *sum* of the i integers in group i, call it σ_i, equals

$$\begin{aligned}\sigma_i &= 2i\Delta_{i-1} + \big(1 + 3 + \cdots + (2i-1)\big) \\ &= 2i\Delta_{i-1} + (\text{the sum of the first } i \text{ odd numbers}) \\ &= 2i\Delta_{i-1} + i^2\end{aligned}$$

By direct calculation, then,

$$\sigma_i = 2i \cdot \frac{i(i-1)}{2} + i^2 = (i^3 - i^2) + i^2 = i^3$$

The proof is now completed by concatenating the rows of the triangle and observing the pattern that emerges:

$$(1) + (3+5) + (7+9+11) + \cdots = 1^3 + 2^3 + 3^3 + \cdots$$

$\square$

Pictorial proof

We now present graphically the relation between sums of perfect cubes and squares of triangular numbers.

This illustration provides a *non-textual* way to understand this result, and it provides a fertile setting for seeking other facts of this type.
For all k,

$$1^3 + 2^3 + \cdots + k^3 = \Delta_k^2$$

Proof

We develop a recurrence that reflects the structure of the previous table.
The intuition of the analogy comes from the previous analysis when the cubes k^3 are decomposed into k squares k^2

Base case.

$$1^3 \;=\; 1 \;=\; \Delta_1^2$$

Proof

We develop a recurrence that reflects the structure of the previous table.
The intuition of the analogy comes from the previous analysis when the cubes k^3 are decomposed into k squares k^2

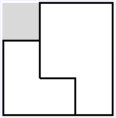
Base case.

$$1^3 = 1 = \Delta_1^2$$

While this first (and obvious) case is enough for the induction, it does not tell us much about the structure of the problem.
Therefore, we consider also the next step $k = 2$:

$$1^3 + 2^3 = 9 = \Delta_2^2$$

1
3 5
7 9 11
13 15 17 19



Inductive hypothesis. Assume that the target equality holds for all $i < k$; i.e.,

$$1^3 \ + \ 2^3 \ + \cdots + \ i^3 \ = \ \Delta_i^2$$

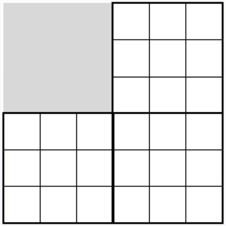
If we go one step further, to incorporate group 3, i.e., the set $\{7, 9, 11\}$, into our pictorial summation process, then we discover that mimicking the previous process is a bit more complicated here. More complicated manipulation required to form the $\Delta_3 \times \Delta_3$ square is a consequence of the odd cardinality of the group-3 set. We must extend our induction for the cases of odd and even k.

Inductive extension for odd k.

$$\Delta_k^2 \ = \ \Delta_{k-1}^2 \ + \ k^3$$

We begin to garner intuition for this extension by comparing the quantities Δ_k^2 and $1 + 2^3 + \cdots + k^3$.

Moving to the pictorial domain, we write k^3 as $k \times k^2$, and we distribute $k \times k$ square blocks around the $\Delta_{k-1} \times \Delta_{k-1}$ square, as shown below for the case $k = 3$.

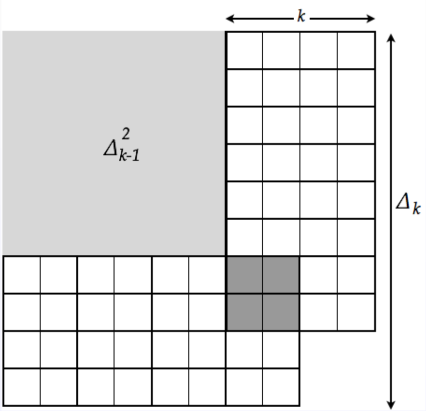


Because k is odd, the small squares pack perfectly since $(k-1)$ is even, hence divisible by 2.
The depicted case depicts pictorially the definition of triangular numbers: $k \cdot \frac{1}{2}(k-1) \ = \ \Delta_{k-1}$.

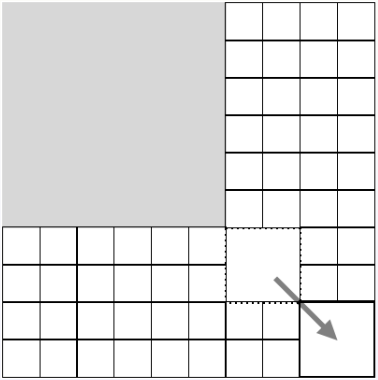
Inductive extension for even k.

The basic reasoning here mirrors that for odd k, with one small difference.

Now, as we assemble small squares around the large square, two subsquares overlap, as depicted below.



We must manipulate the overlapped region in order to get a tight packing around the large square.



Happily, when there is a small overlapping square region, there is also an identically shaped empty square region, as suggested by these two figures.

More details.

Because $(k-2)$ is even, the like-configured square blocks can be allocated to two sides of the initial $\Delta_{k-1} \times \Delta_{k-1}$ square (namely, its right side and its bottom).
The overlap has the shape of a square that measures $\frac{1}{2}(\Delta_k - \Delta_{k-1})$ on a side.
One also sees in the figure an empty square in the extreme bottom right of the composite $\Delta_k \times \Delta_k$ square, which matches the overlapped square identically. This situation is the pictorial version of the equation

$$\Delta_k^2 \; - \; \Delta_{k-1}^2 \;\; = \;\; \frac{1}{4} k^2 \left((k+1)^2 - (k-1)^2 \right) \;\; = \;\; k^3$$

We have thus extended the inductive hypothesis for both odd and even k, whence the result. □