Lecture 1 – Maths for Computer Science Multiple ways for solving a problem Summations

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Lecture notes MoSIG1

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Context and content

The **purpose** of this lecture is to experience multiple ways for solving the same mathematical problem.
Its **goal** is to provide the basis for gaining intuition in proving methods.

We consider the sum of squares as an illustration.

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We consider the sum of squares as an illustration.

- The core analysis: **Sum of squares**
  also called the pyramid numbers
- One step further: the Tetrahedral numbers

Sum of squares: pyramid numbers

Definition:
Sum of the n first squares:
$\square_n = \sum_{k=1}^{n} k^2$

- Let us study various ways to establish and prove the sum of squares.

Preliminary: determine the asymptotic behavior

Rough analysis.

Upper bound

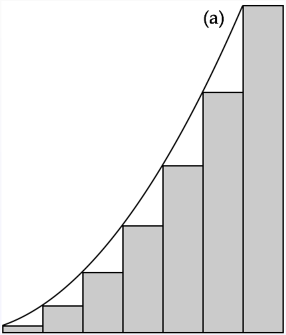
as $k^2 \leq n^2, \forall k \leq n$

$\square_n \leq \sum_{k=1}^{n} n^2 = n^3$

asymptotic behavior (2)

A slightly more precise analysis based on integral leads to:

$\Box_n \leq c\frac{n^3}{3}$



In other words, the summation is in $O(\frac{n^3}{3})$.

asymptotic behavior (3)

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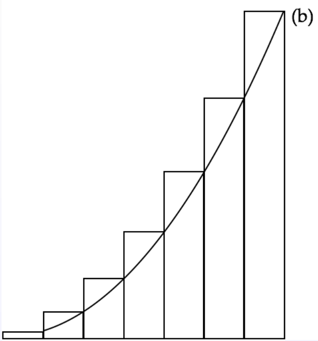
$\square_n \geq c' \frac{n^3}{3}$



It is in $\Omega(\frac{n^3}{3})$, thus, the sum we are looking for is $\Theta(\frac{n^3}{3})$

Method 1: undetermined coefficients

- From the previous asymptotic analysis, we know that:

  $\square_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \alpha_3 n^3$

- we identify the α_i by taking simple values of n

  $\square_0 = \alpha_0 = 0$
  $\square_1 = \alpha_1 + \alpha_2 + \alpha_3 = 1$
  $\square_2 = 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = 5$
  $\square_3 = 3\alpha_1 + 9\alpha_2 + 27\alpha_3 = 14$

Method 1: undetermined coefficients

- Let us solve this linear system.

  $\alpha_1 = 1 - \alpha_2 - \alpha_3$
  $2(1 - \alpha_2 - \alpha_3) + 4\alpha_2 + 8\alpha_3 = 5$
  $3(1 - \alpha_2 - \alpha_3) + 9\alpha_2 + 27\alpha_3 = 14$

  $2\alpha_2 + 6\alpha_3 = 3$
  $6\alpha_2 + 24\alpha_3 = 11$

- After another substitution and some arithmetic manipulations:

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  $2\alpha_2 + 6\alpha_3 = 3$
  $6\alpha_2 + 24\alpha_3 = 11$

- After another substitution and some arithmetic manipulations:

  $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{1}{2}$ and $\alpha_3 = \frac{1}{3}$

  Thus, $\square_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$

Method 2: proving by recurrence

- We need here the expression beforehand and prove it.

Compute the first ranks:

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	9	16	25	36	49	64	81	100
Sn	0	1	5	14	30	55	91	140	204	285	385

Guess the expression (or take it in a book):

$$\square_n = \frac{n(n+1)(2n+1)}{6}$$

Strong induction

- Basis $n = 1$: $\square_1 = \frac{(2\times3)}{6} = 1^2$

Strong induction

- Basis $n = 1$: $\square_1 = \frac{(2\times3)}{6} = 1^2$
- Assume $\square_n = \frac{n(n+1)(2n+1)}{6}$

  Compute $\square_{n+1} = \square_n + (n+1)^2$

  $= (n+1)\frac{n(2n+1)}{6} + (n+1)^2$

  $= (n+1)\frac{2n^2+n+6n+6}{6}$

  $= \frac{(n+1)(n+2)(2n+3)}{6}$

Method 3: perturb the sum

Developing two ways to compute $C_n = \sum_{k=1}^{n} k^3$ allows to express $\square_n$.

1. $C_{n+1} = 1 + \sum_{k=2}^{n+1} k^3$
   $= 1 + \sum_{k=1}^{n} (k+1)^3$
   $= 1 + \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1)$
   $= 1 + C_n + 3\square_n + 3\Delta_n + n$

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2. $C_{n+1} = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n+1)^3 + C_n$
   $= n^3 + 3n^2 + 3n + 1 + C_n$

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Let now equal both expression to deduce $\square_n$.

$$1 + 3\square_n + 3\frac{n^2+n}{2} + n = n^3 + 3n^2 + 3n + 1$$

$$3\square_n = n^3 + 3n^2 + 2n - 3\frac{n^2+n}{2} = n^3 + \frac{3n^2}{2} + \frac{n}{2}$$

Method 4: expand and contract the sum

$\square_n = \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$

$= 1 + (2 + 2) + (3 + 3 + 3) + (4 + 4 + 4 + 4) + ... + (n + n + ... + n)$

Method 4: expand and contract the sum

$\Box_n = \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$

$= 1 + (2 + 2) + (3 + 3 + 3) + (4 + 4 + 4 + 4) + ... + (n + n + ... + n)$

$= (1 + 2 + ... + n) + (2 + 3 + ... + n) + (3 + 4 + ... + n) + ... + n$

Method 4: expand and contract the sum

$\Box_n = \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$

$= 1 + (2+2) + (3+3+3) + (4+4+4+4) + ... + (n+n+...+n)$

$= (1+2+....+n) + (2+3+...+n) + (3+4+...+n) + ... + n$

$= \sum_{k=0}^{n-1} (\Delta_n - \Delta_k)$

$= n.\Delta_n - \sum_{k=1}^{n-1} \Delta_k$

$\Box_n = \frac{n^2(n+1)}{2} - \sum_{k=1}^{n-1} \frac{k^2}{2} - \frac{1}{2}\Delta_{n-1}$

$\Box_n = \frac{n^2(n+1)}{2} - \frac{1}{2}(\Box_n - n^2) - \frac{n(n-1)}{4}$

Method 4: expand and contract the sum

$$\Box_n = \sum_{k=1}^{n} k^2$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{k} k$$

$$= 1 + (2+2) + (3+3+3) + (4+4+4+4) + ... + (n+n+...+n)$$

$$= (1+2+....+n) + (2+3+...+n) + (3+4+...+n) + ... + n$$

$$= \sum_{k=0}^{n-1} (\Delta_n - \Delta_k)$$

$$= n.\Delta_n - \sum_{k=1}^{n-1} \Delta_k$$

$$\Box_n = \frac{n^2(n+1)}{2} - \sum_{k=1}^{n-1} \frac{k^2}{2} - \frac{1}{2}\Delta_{n-1}$$

$$\Box_n = \frac{n^2(n+1)}{2} - \frac{1}{2}(\Box_n - n^2) - \frac{n(n-1)}{4}$$

$$\frac{3}{2}\Box_n = \frac{1}{2}(n^3 + n^2 + n^2 - \frac{n^2-n}{2})$$

$$\Box_n = \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{n}{2})$$

Method 5: semi-graphical proof

- As we already remarked, the sum can be written as: 1, $2+2$, $3+3+3$, etc.
- This is "naturally" represented by triangles of integers
- Compute three rotated triangles as follows:

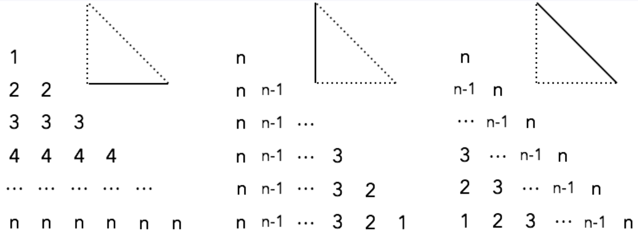


Exhibit an invariant

1						**n**						**n**					
2	2					n	n-1					n-1	n				
3	3	3				n	n-1	…				…	n-1	n			
4	4	4	4			n	n-1	…	3			3	…	n-1	n		
…	…	…	…	…		n	n-1	…	3	2		2	3	…	n-1	n	
n	n	n	n	n	n	n	n-1	…	3	2	1	1	2	3	…	n-1	n

1						n						n					
2	2					**n**	n-1					**n-1**	n				
3	3	3				n	n-1	···				···	n-1	n			
4	4	4	4			n	n-1	···	3			3	···	n-1	n		
···	···	···	···	···		n	n-1	···	3	2		2	3	···	n-1	n	
n	n	n	n	n	n	n	n-1	···	3	2	1	1	2	3	···	n-1	n

1						n						n					
2	**2**					n	**n-1**					n-1	**n**				
3	3	3				n	n-1	…				…	n-1	n			
4	4	4	4			n	n-1	…	3			3	…	n-1	n		
…	…	…	…	…		n	n-1	…	3	2		2	3	…	n-1	n	
n	n	n	n	n	n	n	n-1	…	3	2	1	1	2	3	…	n-1	n

Gather the whole in a single triangle

2n+1					
2n+1	2n+1				
2n+1	2n+1	2n+1			
2n+1	2n+1	2n+1	2n+1		
...	...	...	...	...	
2n+1	2n+1	2n+1	2n+1	2n+1	2n+1

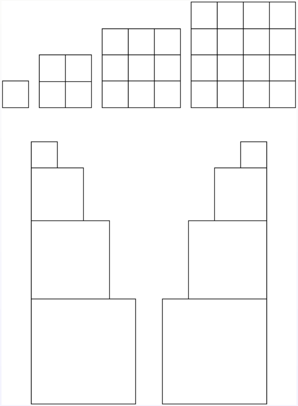
$$
\begin{array}{rcccccc}
 & 1 & & & & & \\
 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
(2n+1) \times & 1 & \cdots & 1 & 1 & & \\
 & 1 & 1 & \cdots & 1 & 1 & \\
 & 1 & 1 & 1 & \cdots & 1 & 1
\end{array}
$$

$$
\begin{array}{lcccccc}
 & 1 & & & & & \\
 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
(2n+1)\,\mathrm{x} & 1 & \cdots & 1 & 1 & & \\
 & 1 & 1 & \cdots & 1 & 1 & \\
 & 1 & 1 & 1 & \cdots & 1 & 1
\end{array}
$$

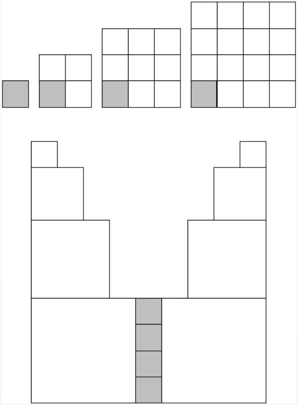
$$3\square_n = (2n+1) \cdot \Delta_n = (2n+1) \cdot \frac{n(n+1)}{2}$$

Method 6: derived graphical proof

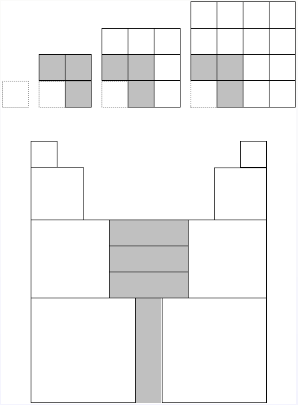
Consider 3 copies of the sum represented by unit squares.



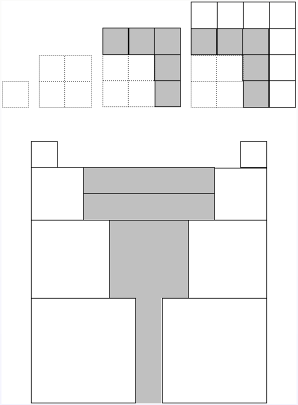
Graphical proof



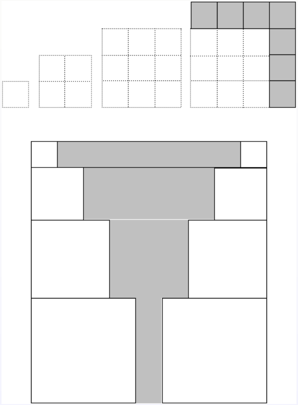
Graphical proof



Graphical proof



Graphical proof



Graphical proof

Conclusion:

- The surfaces of the 3 sums perfectly fits a rectangle.
- The whole area is $2n + 1$ by $\Delta_n = \frac{n(n+1)}{2}$.

Thus, $3\Box_n = \frac{(2n+1)n(n+1)}{2}$

Tetrahedral numbers

Definition:

The sum of the Δ_n is denoted by: $\Theta_n = \sum_{k=1}^{n} \Delta_k$

Tetrahedral numbers

Definition:

The sum of the Δ_n is denoted by: $\Theta_n = \sum_{k=1}^{n} \Delta_k$

- Like for the sum of squares, a way to calculate it is to consider 3 copies of Θ_n and organize them as triangles.

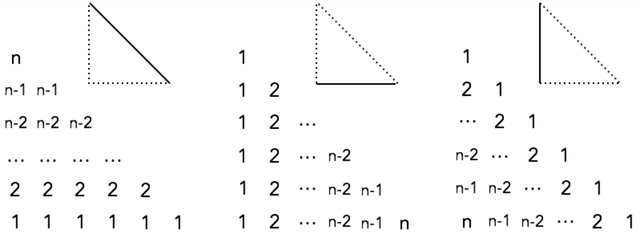


Exhibit an invariant

n					
n-1	n-1				
n-2	n-2	n-2			
…	…	…	…		
2	2	2	2	2	
1	1	1	1	1	1

1					
1	2				
1	2	…			
1	2	…	n-2		
1	2	…	n-2	n-1	
1	2	…	n-2	n-1	n

1					
2	1				
…	2	1			
n-2	…	2	1		
n-1	n-2	…	2	1	
n	n-1	n-2	…	2	1

n					
n-1	n-1				
n-2	n-2	n-2			
…	…	…	…		
2	2	2	2	2	
1	1	1	1	1	1

1					
1	2				
1	2	…			
1	2	…	n-2		
1	2	…	n-2	n-1	
1	2	…	n-2	n-1	n

1					
2	1				
…	2	1			
n-2	…	2	1		
n-1	n-2	…	2	1	
n	n-1	n-2	…	2	1

n	1	1
$n-1$ $\mathbf{n-1}$	1 $\mathbf{2}$	2 $\mathbf{1}$
$n-2$ $n-2$ $n-2$	1 2 $\cdots$	$\cdots$ 2 1
$\ldots$ $\ldots$ $\ldots$ $\ldots$	1 2 $\cdots$ $n-2$	$n-2$ $\cdots$ 2 1
2 2 2 2 2	1 2 $\cdots$ $n-2$ $n-1$	$n-1$ $n-2$ $\cdots$ 2 1
1 1 1 1 1 1	1 2 $\cdots$ $n-2$ $n-1$ n	n $n-1$ $n-2$ $\cdots$ 2 1

Gather the whole in a single triangle

n+2					
n+2	n+2				
n+2	n+2	n+2			
n+2	n+2	n+2	n+2		
...	...	...	...	...	
n+2	n+2	n+2	n+2	n+2	n+2

$$
\begin{array}{ccccccc}
 & 1 & & & & & \\
 & 1 & 1 & & & & \\
 & 1 & 1 & 1 & & & \\
(n+2)\,\mathrm{x} & 1 & \cdots & 1 & 1 & & \\
 & 1 & 1 & \cdots & 1 & 1 & \\
 & 1 & 1 & 1 & \cdots & 1 & 1
\end{array}
$$

$$
\begin{array}{ll}
 & 1 \\
 & 1 \quad 1 \\
 & 1 \quad 1 \quad 1 \\
(n+2) \times & 1 \quad \cdots \quad 1 \quad 1 \\
 & 1 \quad 1 \quad \cdots \quad 1 \quad 1 \\
 & 1 \quad 1 \quad 1 \quad \cdots \quad 1 \quad 1
\end{array}
$$

$$3\Theta_n = (n+2) \cdot \Delta_n = (n+2) \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{2}$$

Another (analytical) way to look at the proof

- The proof is obtained by the double counting Fubini's principle by copying (with a rotation) the basic triangles.

The sum of the first row is equal to $n + 2$.
The second one is equal to $2(n - 1) + 3 + 3 = 2(n + 2)$.

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- The proof is obtained by the double counting Fubini's principle by copying (with a rotation) the basic triangles.

The sum of the first row is equal to $n+2$.
The second one is equal to $2(n-1)+3+3=2(n+2)$.

Let us sum up the elements in row k:
$\Delta_k + \Delta_k + k(n-k+1) = k(k+1) + kn - k^2 + k = k(n+2)$

Thus, the global sum is equal to $(n+2) \times (1+2+...+n)$

Finally, $3\Theta_n = (n+2)\Delta_n$

A first synthesis

We proved some results in this lecture, in particular:

- $Id_n = 1 + 1 + ... + 1 = n$
- $\Delta_n = 1 + 2 + 3 + ... + n = \frac{1}{2} \cdot Id_n \cdot (n + 1)$
- $\Theta_n = \Delta_1 + \Delta_2 + ... + \Delta_n = \frac{1}{3} \cdot \Delta_n \cdot (n + 2)$

A natural question is if we can go further following the same pattern for computing $\sum_{k=1}^{n} \Theta_k$, and so on.

The next family is the *pentatope* numbers (denoted by Π_n), defined as the sum of Θ_k.

More properties

If we write these numbers as polynomials of n, we obtain:

- Rank 1. $Id_n = n$
- Rank 2. $\Delta_n = \frac{1}{2}n(n+1)$
- Rank 3. $\Theta_n = \frac{1}{6}n(n+1)(n+2)$ where $6 = 1 \times 2 \times 3$
- Rank 4. $\Pi_n = \frac{1}{24}n(n+1)(n+2)(n+3)$ where $24 = 1 \times 2 \times 3 \times 4$

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- Rank 4. $\Pi_n = \frac{1}{24}n(n+1)(n+2)(n+3)$ where $24 = 1 \times 2 \times 3 \times 4$
- The next one (rank 5) is $\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4)$

As these numbers are integers
$P(n) = n(n+1)(n+2)(n+3)$ is a multiple of 4!

Exercise

Proving the expectation

- Taking into account the expressions of $Id_n = n$, $\Delta_n = \frac{1}{2}n(n+1)$ and $\Theta_n = \frac{1}{3!}n(n+1)(n+2)$
- Prove: $\sum_{k=1}^{n} \Theta_k = \frac{1}{4!}n(n+1)(n+2)(n+3)$ by an inductive argument on the rank

Coming back on pyramid numbers

- Is there a link between pyramid and tetrahedral numbers?

Coming back on pyramid numbers

- Is there a link between pyramid and tetrahedral numbers?
- Yes!

  There is a link between the two first ranks: Id_n and Δ_n Since $n^2 = \Delta_n + \Delta_{n-1}$

  By summation, we deduce immediately $\Box_n = \Theta_n + \Theta_{n-1}$

The proof follows directly following this definition.

Another property

- Is there a link between triangular and tetrahedral numbers?

Another property

- Is there a link between triangular and tetrahedral numbers?

- Yes!
  Using the expression of Method 4.

$$\square_n = \Delta_n + (\Delta_n - \Delta_1) + (\Delta_n - \Delta_2) + \ldots + (\Delta_n - \Delta_{n-1})$$
$$= n.\Delta_n - \sum_{1 \leq k \leq n-1} \Delta_k$$
$$= n.\Delta_n - \Theta_{n-1}$$

$$\square_n + \Theta_{n-1} = n.\Delta_n$$

This can be shown again using the expanded representation of triangles!

Concluding remarks

We presented in this lecture many ways for solving the same problem.

Take home message:

Everyone can find her/his own method!

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Take home message:

Everyone can find her/his own method!

- The results are interesting and they show the hidden structures of numbers.
- But, more important is the way to solve and to write the proofs.