Maths for Computer Science
Relations

Denis TRYSTRAM
MoSIG1

Sept. 1, 2024

Study of other types of Mathematical objects

- Relations
- logarithms

Equivalence relations

A binary relation ρ is an equivalence relation if the three following properties are fulfilled:

1. Reflexivity: $x\rho x$
2. Symmetry: if $x\rho y$ then $y\rho x$
3. Transitivity: if $x\rho y$ and $y\rho z$ then $x\rho z$

Example

Prove that the following relation between pairs of integers (n_i, m_i):
$(n_1, m_1)\rho(n_2, m_2)$ iff $n_1 + m_2 = n_2 + m_1$ is an equivalence relation

Example

Prove that the following relation between pairs of integers (n_i, m_i):
$(n_1, m_1)\rho(n_2, m_2)$ iff $n_1 + m_2 = n_2 + m_1$ is an equivalence relation

- Intuitively, this relation reflects the geometrical argument that states that the two pairs of points (n_1, m_1) and (n_2, m_2) are equivalent iff the differences $n_1 - m_1$ and $n_2 - m_2$ are equal.
- Draw the picture to get evidence!

Example

Prove that the following relation between pairs of integers (n_i, m_i):
$(n_1, m_1)\rho(n_2, m_2)$ iff $n_1 + m_2 = n_2 + m_1$ is an equivalence relation

- Intuitively, this relation reflects the geometrical argument that states that the two pairs of points (n_1, m_1) and (n_2, m_2) are equivalent iff the differences $n_1 - m_1$ and $n_2 - m_2$ are equal.
- Draw the picture to get evidence!
- Thus, the **equivalence classes** here correspond to straight lines parallel to the first bisectrice

Partitions

Partition is a concept closely linked with binary equivalence relations:

- A partition of a set S and an equivalence relation within S are just two sides of the same medal.
- Both notions are *equivalent*

Order relation

Definition

A binary relation ρ on a set S is a *partial order* if ρ is transitive

Cartesian product

of two sets S and S' (where S' may be the same as S).
It is the set of all ordered pairs of elements whose first coordinate is in S and the second one is in S'.

Common example of use: any binary relations.

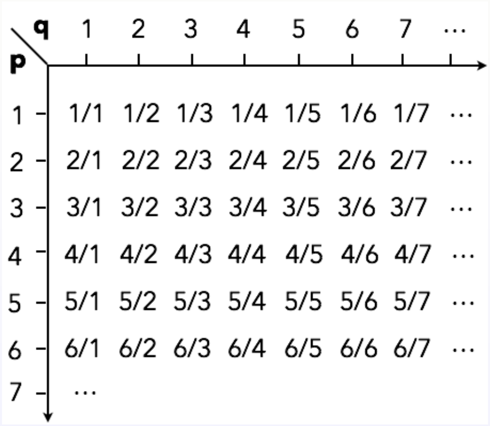
Enumeration of the rationals

Here is a nice illustration of cartesian product over the set N that shows that this set has the same cardinality as Q

- Our goal is to enumerate all the fractions.

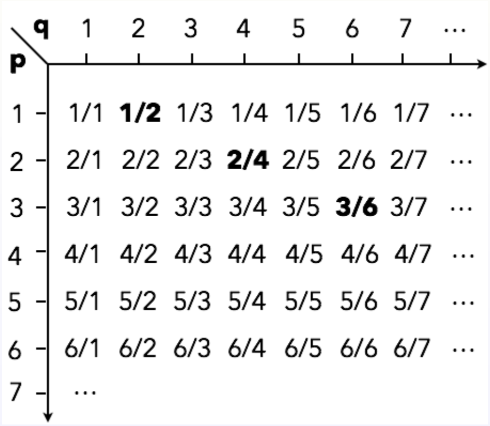
We write all possible fractions in a double-input array (one for the numerator p, one for the denominator q): $\frac{p}{q}$

technically



p \ q	1	2	3	4	5	6	7	⋯
1	1/1	1/2	1/3	1/4	1/5	1/6	1/7	⋯
2	2/1	2/2	2/3	2/4	2/5	2/6	2/7	⋯
3	3/1	3/2	3/3	3/4	3/5	3/6	3/7	⋯
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7	⋯
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7	⋯
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7	⋯
7	⋯							

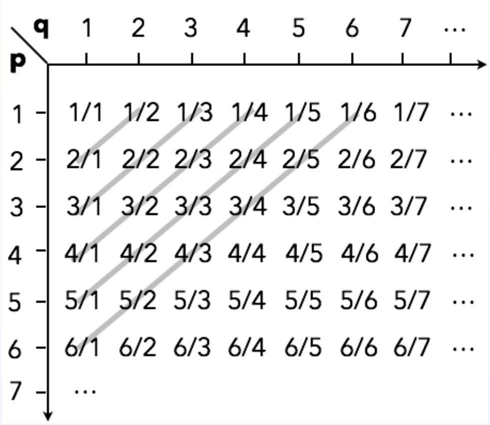
Equivalence classes



p \ q	1	2	3	4	5	6	7	⋯
1	1/1	**1/2**	1/3	1/4	1/5	1/6	1/7	⋯
2	2/1	2/2	2/3	**2/4**	2/5	2/6	2/7	⋯
3	3/1	3/2	3/3	3/4	3/5	**3/6**	3/7	⋯
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7	⋯
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7	⋯
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7	⋯
7	⋯							

- Some rationals may be reduced to an irreductible fraction.

The way to enumerate



p \ q	1	2	3	4	5	6	7	⋯
1	1/1	1/2	1/3	1/4	1/5	1/6	1/7	⋯
2	2/1	2/2	2/3	2/4	2/5	2/6	2/7	⋯
3	3/1	3/2	3/3	3/4	3/5	3/6	3/7	⋯
4	4/1	4/2	4/3	4/4	4/5	4/6	4/7	⋯
5	5/1	5/2	5/3	5/4	5/5	5/6	5/7	⋯
6	6/1	6/2	6/3	6/4	6/5	6/6	6/7	⋯
7	⋯							

- There is a one-to-one correspondence (along the diagonals)
- This proves that N and Q have the same cardinatilty.

Some mechanisms that underlie mathematical reasoning

Formalizing hypotheses, decomposing arguments into steps, invoking logical inference, and writing proofs (that is our ultimate goal).

- What is the essence of logical reasoning? of logical argumentation?
- What does it mean to say that one proposition *implies* another?
- When has one established that two propositions are "equivalent", in the sense that logical arguments cannot distinguish them?

Basics of reasoning: Syllogisms

Form of reasoning introduced by Aristotle

- It is a logical reasoning that links at least three propositions: two or more, called "premises", lead to a "conclusion".
- These propositions are generally expressed with unary predicates only, and therefore belong to first-order monadic logic.

[1] change "mortal" by "kleptoman"

Basics of reasoning: Syllogisms

Form of reasoning introduced by Aristotle

- It is a logical reasoning that links at least three propositions: two or more, called "premises", lead to a "conclusion".
- These propositions are generally expressed with unary predicates only, and therefore belong to first-order monadic logic.

Example

Main premise: **All men are mortal**
Secondary premise: **Socrates is a man**
Conclusion: **Socrates is mortal**

Attention: Don't confuse validity with truth[1]!

[1] change "mortal" by "kleptoman"

Algebra of Propositional Logic

- The **propositions** are the basic objects (syntactic), they are assertions that can be true or false.
- Propositional logic is the restriction without the quantifiers.

Example: "the sky is pink".

Similarly to Set Theory, we can combine propositions in more complex ones:

- If (the sky is dark) and (I must go to the Maths class at UGA) then (I take an umbrella).

The connective that links the various components are obtained by some operations **not**, **or**, **and**, **xor**, **implies**, ...
The corresponding system put in operations with boolean variables is an **Algebra**

- The *commutativity* of union and intersection:

$$\begin{aligned} S \cup T &= T \cup S \\ S \cap T &= T \cap S \end{aligned}$$

- The *distributivity* of either of union and intersection:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T) \tag{1}$$
$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T) \tag{2}$$

  Note that arithmetic (of numbers) has an analogue of Eq. (2)—with multiplication playing the role of intersection and addition playing the role of union—but it does not *have an analogue of Eq. (1).*

- The *idempotence* of complementation:

$$\overline{\overline{S}} = S$$

Operations

What distinguishes Propositional Logic from more general mathematical logic is the absence of quantifiers (THERE EXISTS, FOR ALL, etc.).

The algebra that underlies Propositional Logic uses operations that are reminiscent of the set-theoretic operations to combine simple assertions into complex ones

- **if** "bananas are ripe" **and** "you are hungry" **then** "buy bananas"
- **either** "the grass is green" **or** "the ocean is calm"
- Two special propositions – the constants of the algebra – are denoted TRUE and FALSE.
  They are *intended* to represent factual truth and falsehood, but they are *defined* by the way they interact with other propositions.

(i) The *unary* logical connective

- NOT: **negation** ($\neg$) (Set-theoretic analogue: *complementation*). Two shorthand notations for "NOT P" have evolved:
    - the prefix-operator $\neg$, as in "$\neg P$"
    - the overline-operator, as in "$\overline{P}$"

  Whichever notation one uses, the defining properties of negation are encapsulated in the following equations.

$$[\neg \text{TRUE} = \text{FALSE}] \quad \text{and} \quad [\neg \text{FALSE} = \text{TRUE}]$$

(ii) The *binary* logical connectives

OR: **disjunction** $(\vee)$ (Set-theoretic analogue: *union*).

The operation OR – which is also called *logical sum* – is usually denoted by the infix-operator $\vee$;
the operation's defining properties are encapsulated as follows.

$[[P \vee Q] = \text{TRUE}]$ if, and only if, $[P = \text{TRUE}]$ or $[Q = \text{TRUE}]$ or both.

Note that, as with union, logical OR is *inclusive:* The assertion

$[P \vee Q]$ is TRUE

is true when *both* propositions P and Q are true, as well as when only one of them is. Because such inclusivity does not always capture one's intended meaning, there is also an *exclusive* version of disjunction, as we see next.

XOR: **XOR** ($\oplus$) (Set-theoretic analogue: *disjoint union*).
The operation *exclusive or* is a version of disjunction that does *not* allow both disjuncts to be true simultaneously. It is usually denoted by the infix-operator $\oplus$; the operation's defining properties are encapsulated as follows.

$[[P \oplus Q] = \text{TRUE}]$ if, and only if, $[P = \text{TRUE}]$ or $[Q = \text{TRUE}]$ *but not* b

We emphasize the distinction between $\vee$ and $\oplus$, the (respectively) inclusive and exclusive versions of disjunction, by remarking that the assertion

$$[P \oplus Q] \text{ is TRUE}$$

is *false* when both propositions P and Q are true.

AND: **conjunction** $(\wedge)$ (Set-theoretic analogue: *intersection*).
The operation AND – which is also called *logical product*—is usually denoted by the infix-operator $\wedge$; the operation's defining properties are encapsulated as follows.

$[[P \wedge Q] = \text{TRUE}]$ if, and only if, *both* $[P = \text{TRUE}]$ and $[Q = \text{TRUE}]$.

IMPLIES: **logical implication** ($\Rightarrow$) (Set-theoretic analogue: *subset*).
The logical operation IMPLIES is often called *conditional*

$$[[P \;\Rightarrow\; Q] = \textsc{true}] \quad \text{if, and only if,}$$

$$[[\neg P] = \textsc{true}] \text{ (inclusive) or } [Q = \textsc{true}].$$

- *If proposition P is false, then it implies* every *proposition.*
- *If proposition Q is true, then it is implied by* every *proposition.*

The *semantic completeness* of the Propositional Calculus as a logical system is a consequence of the fact that we are able to view the expressions of the Calculus as Boolean functions, in the following way.

- As we examine an expression in the Calculus, the only information we need about the propositions which appear in the expression is the array of truth-values for the propositions.

If we tabulate how the truth-values of propositions combine under the logical operators that interconnect them in the expression, then we remark immediately how the expressions can be viewed as *functions of binary tuples*, where the arity of the functions is the number of propositional variables.

Using this viewpoint, the tables reproduce the definitions of the main logical operators , viewed as functions within the space of truth-values. Each propositional variable is instantiated with all of its possible truth-values, TRUE and FALSE – which we denote here, for convenience, by 1 and 0, respectively.

P	Q	$P \vee Q$	$P \oplus Q$	$P \wedge Q$	$P \Rightarrow Q$	$P \equiv Q$
0	0	0	0	0	1	1
0	1	1	1	0	1	0
1	0	1	1	0	0	0
1	1	1	0	1	1	1

Every Propositional expression is a binary function, so we can pass back and forth between logical and functional/operational terminology.
This ability affords us very simple definitions of two important concepts that are somewhat harder to define in purely logical terms.

- Tautology.
  - *Mathematical formulation:*
    A Propositional expression is a *tautology* iff its corresponding function is the constant function $F(x) \equiv 1$.
  - *Logical formulation:*
    A Propositional expression is a *tautology* iff it evaluates to $\textsc{true}$ under every instantiation of truth-values for its Propositional variables.

- Satisfiable expression.
  - *Mathematical formulation:*
    A Propositional expression is *satisfiable* iff its corresponding function has 1 in its range; i.e., iff there is an argument x such that $F(x) = 1$.
  - *Logical formulation:*
    A Propositional expression is *satisfiable* iff there exists an instantiation of truth-values for its Propositional variables under which the expression evaluates to TRUE.

This last notion is a foundation of complexity theory (SAT problem).

IFF: **logical equivalence** ($\equiv$) (Set-theoretic analogue: *set equality*). The final logical operation that we shall discuss is known by many names, including *logical equivalence* and *biconditional*, as well as *if and only if* and its shorthand IFF. It is usually denoted via one of the following two infix-operators: $\equiv$ or $\Leftrightarrow$; the operation's defining properties are encapsulated as follows.

$$[[P \;\equiv\; Q] = \text{TRUE}] \quad \text{if, and only if}$$

$$[[P \;\Rightarrow\; Q] = \text{TRUE}] \quad \text{and } [[Q \;\Rightarrow\; P] = \text{TRUE}].$$

algebraic closure

- Let C be a (finite or infinite) collection of sets.
- Let S and T be elements of collection C.
- Let $\circ$ be an operation on sets – so that $S \circ T$ is a set.

We say that collection C is *closed* under the operation $\circ$ if whenever sets S and T (which could be the same set) both belong to C, the set $S \circ T$ also belongs to C.