Dice and other Stories

Abstract Model of Randomness

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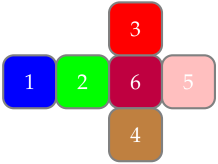
MOSIG 1 Mathematics for Computer Science



September 2024

UGA Université Grenoble Alpes

PROBABILITY AND COMPUTER SCIENCE

Modeling

- Data modeling : text compression (entropy), algorithm analysis,...
- Performance evaluation : workload description, users profile,...

Randomization

- Probabilistic method
- Random based algorithms (cryptography)
- Simulation of systems

MR. AND MRS. SMITH

Exercice 1a

Mr. and Mrs. Smith have two children, one is a boy, what is the probability that the other is a girl ?

Mr. and Mrs. Smith

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Mr. and Mrs. Smith have two children, the elder is a boy, what is the probability that the younger is a girl ?

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Exercice 1c

What are the difficulties to solve such problems ?

Hint

Mr and Mrs Smith (simulation)

```python
import random
random.seed(42)

# Mr and Mrs Smith have 2 kids one is a boy.
# What is the probability that the other is a girl ?
def birth():
    return random.choice(['B', 'G'])

def family(size=2):
    return [birth() for _ in range(size)]

def sample_families(size = 2, sample_size = 1000):
    return [family(size) for _ in range(sample_size)]

## informal tests
s_f = sample_families(2,100)
s_f_b = [x for x in s_f if 'B' in x]
s_f_bg = [x for x in s_f_b if 'G' in x]
frequency = len(s_f_bg) / len(s_f_b)
print(frequency)
```

Pascal and Chevalier de Méré discussion (simplified)

A Dice Game

- bet 1
- throw two dices and sum the results
- if the result is 11 or 12 you earn 11 (including your bet)
- if not you loose your bet.

Exercice 2a

The Chevalier de Méré says "playing this game a sufficiently long time and I'll get a fortune" and Pascal argues the contrary. Who is wrong and what were the two arguments ?

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Exercice 2b

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```
import random
random.seed(42)

# Chevalier de Méré's problem
def dice(faces=6):
    return random.randint(1,6)

def play(faces=6, sum=11):
    return (dice() + dice()) >= sum

def game(number_play=100):
    return [play() for _ in range(number_play)]

def average_gain(g):
    gain = 0
    for b in g:
        gain = gain + 10 if b else gain - 1
    return gain/len(g)

## informal tests
print(average_gain(game(1000)))
```

THE MONTY HALL PROBLEM

A TV show problem

There are 3 closed doors beside one there is a magnificent car, beside the two others nothing.

- TV host : Please choose one door. As example you choose door 2.
- TV host : I want to help you. I open one of the remaining door with nothing. For example he opens door 1.
- TV host : in fact you could modify your first choice, do you change your initial decision of choosing door 2.
- As example you decide to change and you open door 3. You win if the car is beside.

Exercice 3a

What is a good strategy : change or not your initial decision ?

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What is a good strategy : change or not your initial decision ?

Exercice 3b

What are the difficulties to solve such problems ?

Wikipedia reference

The Monty Hall Problem (simulation)

```
# Monty Hall problem
def game(strategy):
    config = ['*','_','_']
    random.shuffle(config)
    player = random.randint(0,2)
    if strategy == 0: # strategy keep
        return config[player] == '*'
    else: # strategy change
        return config[player] != '*'
            # bad initial choice for the player
            # the TV entertainer must open the empty door
            # among the two doors the remaining is winning

def experiment(sample_size=1000, strategy=0):
    sample = [game(strategy) for _ in range(sample_size)]
    return sample.count(True)/len(sample)

## informal tests
sample_size = 100000
print("strategy keep, gain probability ",experiment(sample_size,0))
print("strategy change, gain probability ",experiment(sample_size,1))
```

THE CONTROL OF DEMOGRAPHY

In some country, the government try to control the number of births in the country. There are several strategies.

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Are there more male births than female on average ?

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Families are allowed to have a first child, if she is a girl they could have a second one.
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Exercice 4c

Families are allowed to have children, until they get a boy.
Are there more male births than female on average ?

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Families are allowed to have children, until they get a boy.
Are there more male births than female on average ?

Exercice 4d

What are the difficulties to solve such problems ?

THE CONTROL OF DEMOGRAPHY (SIMULATION)

```python
# Natality control
def birth():
    return random.choice(['B', 'G'])

def family(size=None):
    i = 1
    result = [birth()]
    while result[-1] == 'G' and ((size == None) or (i < size)):
        i = i + 1
        result.append(birth())
    return result

def sample_families(size = 2, sample_size = 1000):
    return [family(size) for _ in range(sample_size)]

## informal tests
s_f = sample_families(2,100)
boys = sum([x.count('B') for x in s_f])
girls = sum([x.count('G') for x in s_f])
print(boys/(boys + girls))
```

Abstract representation

Consider now a set Ω, a set $\mathcal{A}$ of parts of Ω is called a σ**-field** if it satisfies the following properties :

1. $\Omega \in \mathcal{A}$;
2. If $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$ (the complement of A in Ω is in $\mathcal{A}$);
3. Let $\{A_n\}_{n\in\mathbb{N}}$ a denumerable set of element of $\mathcal{A}$ then

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A};$$

   (σ-additivity property)

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Interpretation

The set Ω models the real world, which is impossible to capture with all of its complexity. Consequently we observe the reality with measurement tools and get partial information on it. An *event* is a fact we could observe on the physical situation. It assumes the existence of an experience that produces the event which is observable.

Probability

The idea of probability is to put some real value on events, then the probability function is defined on the set of events and associate to each event a real in $[0,1]$.

Basic Axioms

$$\begin{array}{rccc} \mathbb{P}: & \mathcal{A} & \longrightarrow & [0,1]; \\ & A & \longmapsto & \mathbb{P}(A). \end{array}$$

It verifies the following rules :

1. $$\mathbb{P}(\Omega) = 1;$$
2. If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint events (for all (i,j), $A_i \cap A_j = \emptyset$) then
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For a **finite set** $\Omega = \{a_1, \cdots, a_n\}$ and $\mathcal{A}$ the set of all subsets of Ω, the probability is entirely defined from the probability of the elements of Ω, $\mathbb{P}(\{a_i\})$.

$$\text{For } A \in \mathcal{A}, \text{ we have } \mathbb{P}(A) = \sum_{a\in A} \mathbb{P}(\{a\}).$$

PROBABILITY (2)

Probability properties

Let A and B events of Ω :

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Interpretation

The semantic of a probability measure is related to experimentation. Consequently it supposes that we can repeat infinitively experiments in the same conditions. Then the probability of an event (observable) A is the abstraction of the proportion that this event is realized in a large number of experiments. Consequently the probability is an ideal proportion, assuming that we could produce an infinite number of experiments and compute the asymptotic of frequencies.

CONDITIONAL PROBABILITY

Consider B such that $\mathbb{P}(B) > 0$. The conditional probability of an event A knowing B,

$$\mathbb{P}(A|B) \stackrel{def}{=} \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

defines a new probability measure on the set of event $\mathcal{A}$ (check it as an exercise).

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Interpretation

The meaning of conditional probability comes from the fact that we could observe reality through several measurement instruments. The conditional probability considers external information (event) which is given a-priori.

Conditional Probability (2)

The law of total probability (theorem)

Consider a partition of Ω in a countable set of observable events $\{B_n\}$ ($\mathbb{P}(B_n) > 0$).

$$\Omega = B_1 \cup B_2 \cup \cdots \cup B_n \cup \cdots \text{ and for all } i \neq j,\ B_i \cap B_j = \emptyset$$

The law of total probability states that for all $A \in \mathcal{A}$

$$\mathbb{P}(A) = \sum_n \mathbb{P}(A|B_n)\mathbb{P}(B_n).$$

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Interpretation

The law of total probability explains that if we have a set of disjoint alternatives, we could compute the probability of an event by computing its probability knowing each alternative and then combine all of them with the weight (probability) of each alternative.

INDEPENDENCE

Two events A and B are independents if and only if they satisfy

$$\mathbb{P}(A \cap B) = \mathbb{P}(A).\mathbb{P}(B).$$

This is rewritten, assuming $\mathbb{P}(B) > 0$

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Interpretation

Independence is related to the causality problem. If two events are not independent we could suspect a hidden relation between them, then an event could be the "cause" of the other. On the other side two events are independent if in the observed phenomenon there are no possible relations between the events.

Random Variables

Limits of the arbitrary set approach

- Ω is complex, too complex (related to an experimental procedure)

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Abstraction of the real world by a mapping (random variable)

$$\begin{array}{rcl} X : \Omega & \longrightarrow & E \\ \omega & \longmapsto & X(\omega) \end{array}$$

such that event

$$\{X \in B\} \triangleq \{\omega \in \Omega \text{ such that } X(\omega) \in B\} \in \mathcal{A},$$

Standard description of the σ-field

- Generated by singletons for discrete values (all subsets are events)
- Generated by intervals (Borel σ-fields) for continuous sets (as $\mathbb{R}$, $\mathbb{R}^n$...)

Law (or probability distribution) of a random variable

$$\mathbb{P}(X \in B) = \mathbb{P}(\{\omega \in \Omega \text{ such that } X(\omega) \in B\})$$

A modeling example : result of a dice throw

Old fashion

- $\Omega = \{1, 2, 3, 4, 5, 6\}$ (rough simplification of reality)
- the events are all the subsets of Ω
- the probability law is uniform on all singletons

A MODELING EXAMPLE : RESULT OF A DICE THROW

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Naming the randomness

- Ω experiment (highly complex)
- $\mathcal{A}$ a σ-field on Ω (highly complex)
- $\mathbb{P}$ a probability on $\mathcal{A}$

Assumption

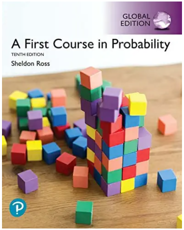
- model the random experiment by a random variable X
- values of X are $\{1, \ldots, 6\}$
- and probability law uniform that is $\mathbb{P}(X = i) = \frac{1}{6}$

Synthesis

Global picture

- Reality is hard to capture with common language
- Formal language of probability
- Algebraic rules
  - σ-algebra of events
  - independence and conditional probabilities
- Interpretation

REFERENCES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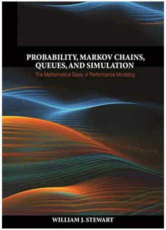
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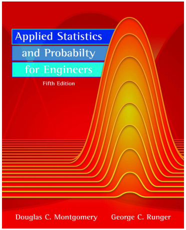
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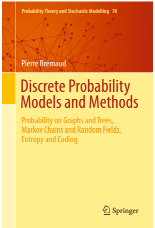
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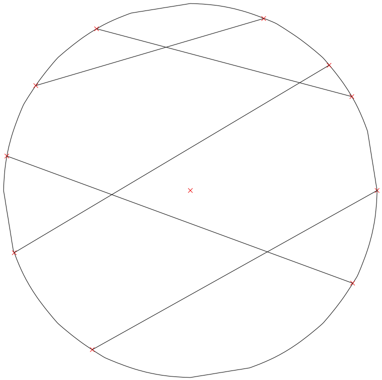
GENERATION OF GEOMETRICAL OBJECTS

Joseph Bertrand : generate a random chord



Compute the probability that the length of the chord is greater than the length of the side of an equilateral triangle inscribed in the circle.

Generation of geometrical objects

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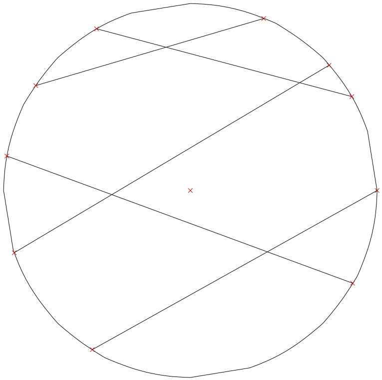

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Alternatives

$$p = \frac{1}{2} \quad p = \frac{1}{3} \quad p = \frac{1}{4}$$

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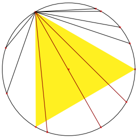
Circle



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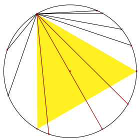

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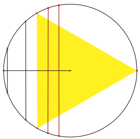
Rays



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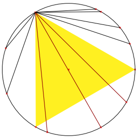

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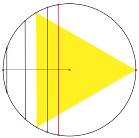

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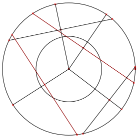
Disc



$$p = \frac{1}{4}.$$

JOSEPH BERTRAND (1822-1900)

Joseph Louis François Bertrand, habituellement appelé Joseph Bertrand, né le 11 mars 1822 à Paris, mort le 3 avril 1900 à Paris, était un mathématicien, historien des sciences et académicien français.

Enfant prodige, à onze ans il suit les cours de l'École Polytechnique en auditeur libre. Entre onze et dix-sept ans il obtient deux baccalauréats, une licence et le doctorat ès sciences avec une thèse sur la théorie mathématique de l'électricité, puis est admis premier au concours d'entrée 1839 de l'École Polytechnique. Il est ensuite reçu au concours de l'agrégation de mathématiques des facultés et premier au premier concours d'agrégation de mathématiques des lycées avec Charles Briot, ainsi qu'à l'École des mines. Il fut professeur de mathématiques au lycée Saint-Louis, répétiteur, examinateur puis professeur d'analyse en 1852 à l'École polytechnique et titulaire de la chaire de physique et mathématiques au Collège de France en 1862 en remplacement de Jean-Baptiste Biot.
En 1845, en analysant une table de nombres premiers jusqu'à 6 000 000, il fait la conjecture qu'il y a toujours au moins un nombre premier entre n et 2n-2 pour tout n plus grand que 3. Tchebychev a démontré cette conjecture, le postulat de Bertrand, en 1850.
Pour l'étude de la convergence des series numériques, il mit au point un critère de comparaison plus fin que le critère de Riemann.

$$\sum \frac{1}{n^\alpha \log n^\beta} \text{ converge ssi } (\alpha, \beta) \geqslant (1, 1).$$

Hints for solutions Exercise1

Exercice 1a

Mr. and Mrs. Smith have two children, one is a boy, what is the probability that the other is a girl ?

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Girl	X	

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Mr. and Mrs. Smith have two children, the elder is a boy, what is the probability that the younger is a girl ?

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Exercice 1c

What are the difficulties to solve such problems ?

return

PASCAL AND CHEVALIER DE MÉRÉ DISCUSSION (SIMPLIFIED) HINTS

A Dice Game

- bet 1
- throw two dice and sum the results
- if the result is 11 or 12 you earn 11 (including your bet)
- if not you loose your bet.

Exercice 2a

The Chevalier de Méré says "playing this game a sufficiently long time and I'll get a fortune" and Pascal argues the contrary. Who is wrong and what were the two arguments ?

Pascal and Chevalier de Méré discussion (simplified) Hints

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Intuition

2nd \ 1st	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Probability to win : $\frac{3}{36} = \frac{1}{12}$
- Expected gain : $11 \times \frac{1}{12} + 0 \times \frac{11}{12} - 1 = -\frac{1}{12}$
- On average you loose $\frac{1}{12}$ per game

FORMAL PROOF

Step 1 : The Model (and the question)

Denote by X (resp. Y) the random variable representing the result of the first (resp. second) dice.
Statistical Hypothesis: X and Y have the same probability law, with a uniform distribution on $\{1, 2, 3, 4, 5, 6\}$ and are independent. Denote by $S = X + Y$.
Question : Compute the probability that S is 11 or 12.

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Step 2 : Answer the question

We compute the law of S, using some algebra, for $i \in \{2, \cdots, 12\}$

$$\begin{aligned}
\mathbb{P}(S=i) &= \mathbb{P}(X+Y=i) = \sum_k \mathbb{P}(X=k, Y=i-k) \text{(rule of sum of disjoint subsets)} \\
&= \sum_k \mathbb{P}(X=k)\mathbb{P}(Y=i-k) \text{(independence of } X \text{ and } Y\text{)} \\
& \quad \text{applying to } i = 11 \text{ and } i = 12 \\
\mathbb{P}(S=11) &= \mathbb{P}(X=5)\mathbb{P}(Y=6) + \mathbb{P}(X=6)\mathbb{P}(Y=5) = \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} = \frac{2}{36} \\
\mathbb{P}(S=12) &= \mathbb{P}(X=6)\mathbb{P}(Y=6) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}
\end{aligned}$$

Step 3 : Interpretation

The average gain considering a large number of games is $-\frac{1}{12}$

UGA Université Grenoble Alpes

QUESTIONS

Uniformity problem

The modeling error was to suppose that the result of the sum is uniformly distributed as the two dice are. To help Chevalier de Méré, could you build two different biased dice (both faces are $\{1, 2, 3, 4, 5, 6\}$) so that the result of the sum is uniformly distributed.

Follow the 3 steps.

Elements of proof

Step 1 : Modeling

Model the result of the first dice (resp second) by a random variable X (resp Y) with value in $\{1, 2, 3, 4, 5, 6\}$. The dices are biased so we define the probability law for each as $p_i = \mathbb{P}(X = i)$ and $q_i = \mathbb{P}(Y = i)$ for $i \in \{1, 2, 3, 4, 5, 6\}$.
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Step 2 : Analysis of the formal model

Th probability of each cell should be computed

Y \ Y		1	2	3	4	5	6
		p_1	p_2	p_3	p_4	p_5	p_6
1	q_1	2	3	4	5	6	7
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Convolution of the probability distributions

Probabilities (using independence)

i	$\mathbb{P}(X+Y=i)$
2	p_1q_1
3	$p_1q_2 + p_2q_1$
4	$p_1q_3 + p_2q_2 + p_3q_1$
5	$p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1$
6	$p_1q_5 + p_2q_4 + p_3q_3 + p_4q_2$ $+p_5q_1$
7	$p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3$ $+p_5q_2 + p_6q_1$
8	$p_2q_6 + p_3q_5 + p_4q_4 + p_5q_3$ $+p_6q_2$
9	$p_3q_6 + p_4q_5 + p_5q_4 + p_6q_3$
10	$p_4q_6 + p_5q_5 + p_6q_4$
11	$p_5q_6 + p_6q_5$
12	p_6q_6

Elements of proof (2)

The question

We have to find the unknown $p_1, \cdots, p_6$ and $q_1, \cdots, q_6$ such that for all i

$$\mathbb{P}(X + Y = i) = \frac{1}{11}$$

with the constraints

$$0 \leqslant p_i, q_i \leqslant 1 \text{ and } p_1 + \cdots + p_6 = q_1 + \cdots + q_6 = 1 \text{ (probability)}$$

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Some algebra and analysis

Explain that p_1, q_1, p_6, q_6 are strictly positive
$q_1 = \frac{1}{11p_1}$ and $q_6 = \frac{1}{11p_6}$

$$\begin{aligned}
\mathbb{P}(X+Y=7) &= p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \\
&\geqslant p_1q_6 + p_6q_1 = \frac{1}{11}\left(\frac{p_1}{p_6} + \frac{p_6}{p_1}\right) \\
&> \frac{1}{11} \text{ because either } \frac{p_1}{p_6} \text{ or } \frac{p_6}{p_1} \text{ is strictly greater than 1}
\end{aligned}$$

Elements of Proof (3)

Step 3 : Interpretation

Following the constraints we deduce that $\mathbb{P}(X + Y = 7) > \frac{1}{11}$, so it is impossible to biase the dice such that the sum follow a uniform distribution.

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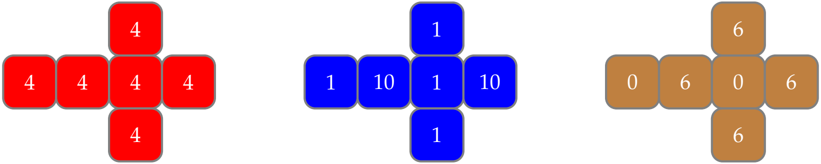
4. Is it possible to change the values on the faces of the dice so that the sum follow the same distribution as the sum of two regular dice ?

Another game



Rules

- Choose one of the dice
- I choose another one
  - Throw your dice as I do for my dice, the best score wins the play
  - Repeat the play
  - until some end

analyse this game